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INVESTIGATING GENERALIZED HYPERGEOMETRIC FUNCTIONS AND THEIR RELATIONS WITH K-FUNCTIONS IN ONE AND TWO VARIABLES

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ABSTRACT

We explore the features and relationships of Generalized Hypergeometric Functions (GHFs) in one and two variables, delving into their unpredictable domain and their noteworthy correlations with K-Functions. GHFs are versatile numerical developments that are well-known for their flexible applications in many fields of research and design. In this paper, we characterize more generalized hypergeometric k-functions using an impressive example of Wright hypergeometric capacity. The basic representation, differential features, touching relations, and differential recipes of the generalized hypergeometric k-functions 2R1, k(a, b; c; τ; z) $(k > 0)$ are somewhat outlined. The aim of this investigation study is to find the neigh boring capability relations for *k*-hypergeometric functions with one boundary and, additionally, to obtain adjoining capacity relations for two boundaries, with reference to Gauss's fifteen bordering capability relations for hypergeometric functions. During this exploratory work, we find the touching capability relations for both cases up to the point where another boundary $k>0$. In the case where $k\to1$, the touching capability relations for *k*-hypergeometric functions are clearly Gauss coterminous capability relations.

KEYWORDS: Investigating, Generalized Hypergeometric Functions, K-Functions, One, Two Variables

1. INTRODUCTION

Mathematicians and academics have been interested in the area of remarkable functions and numerical analysis for a while now. K-Functions and Generalized Hypergeometric Functions (GHFs) are two particularly robust and versatile numerical tools among the many remarkable functions that have found applications in various fields of research and design. These functions have important connections with numerous numerical designs, such as complex analysis, combinatorics, and number theory, and are rich in their numerical intricacy. We explore the space of Generalized Hypergeometric Functions and their complex relationships with K-Functions in one and two variables in this analysis.

The image ${}_pF_q$ represents Generalized Hypergeometric Functions, a class of remarkable functions that are solutions for a particular type of differential condition called the hypergeometric differential condition. These functions can be described in terms of infinite series, and their proportions to gamma functions remain uncertain. The use of Generalized Hypergeometric Functions has been influential in many fields of physical research, design, and science. They are applied to problems in several domains such as factual mechanics, quantum mechanics, and the exceptional relativity theory.

On the other hand, K-Functions provide an additional configuration of exceptional functions that combine the more expansive Wright hypergeometric capability with the traditional Kummer's blended hypergeometric capability. These functions have an intricate and confusing structure, and understanding them is crucial for understanding various complex analyses and basic modifications.

Our analysis aims to shed light on the important relationships that exist between K-Functions and Generalized Hypergeometric Functions in both one and two variables. This study is motivated by the desire to uncover hidden numerical relationships and broaden our understanding of the exceptional quality and practicality of these special functions.

This project will explore the logical characteristics, assembly, and series representations of these functions, enabling us to study their behaviour and interactions with other numerical materials. Additionally, we will examine the useful relationships, fundamental explanations, and modifications that connect these remarkable functions, providing insight into their flexible uses in critical thinking and numerical demonstration.

Our goal in doing this analysis is not only to broaden our understanding of Generalized Hypergeometric Functions and K-Functions but also to highlight their significance within the broader context of science and its applications. The bewildering web of connections we unearth will serve as an exhibit of the class and power of exceptional functions, illuminating their role as fundamental tools in the fields of science and mathematics.

2. LITERATURE REVIEW

M. Abdalla and M. Hidan (2021) address the k-simple of Gauss hypergeometric functions that are created with the Hadamard item. They explore the characteristics and applications of these functions, which are more generalized forms of hypergeometric functions, in this review. These kinds of functions are fundamental to quantifiable mechanics, numerical material science, and several design applications. The authors deepen our understanding of these functions' numerical characteristics

and pave the path for potential uses in a variety of domains by examining how these functions evolve using the Hadamard item.

The study by Nisar, Rahman, Mubeen, and Arshad (2017) looks at basic equations, such as the generalized k-Bessel capability. With applications in quantum mechanics, signal processing, and wave theory, the k-Bessel capacity is a special kind of capability. The developers increase the usefulness of this skill in numerical and mathematical analysis by identifying new required recipes. Their work advances the numerical methods and instruments used in several branches of research and design.

Yilmaz, Aktaş, and Tasdelen (2020) study creating relations with the k-partial subsidiary and the k-simple of Appell functions. A class of exceptional functions called Appell functions is well-known for their use in solving differential conditions in numerical material science and design. Through the k-fragmentary subsidiary, the authors provide producing relations and examine the k-simple of these functions. This work improves our understanding of these remarkable functions' features and uses in fragmented mathematics and related topics. The study conducted by Agarwal and colleagues (2018) investigates fragmented kinetic situations, encompassing the generalized k-Bessel capability. Numerous applications of fragmentary kinetic conditions can be found in numerical material science, design, and other logical fields. The Sumudu change is employed by the authors to examine these conditions, providing a valuable tool for addressing and analyzing partial differential problems. Their research advances our understanding of partial analytics and its uses.

The note by Jana, Maheshwari, and Shukla (2019) discusses the extended hypergeometric capacity. A class of exceptional functions known as hypergeometric functions has many uses in numerical analysis, numerical physical research, and other domains. The authors provide significant insights and information regarding the extended hypergeometric capability, enhancing understanding of its characteristics and potential uses.

The study of Jaeyeong and Heo (2020) focuses on crucial depictions for GKZ hypergeometric functions using MELLIN-Barnes. Advanced exceptional functions known as GKZ hypergeometric functions are used in combinatorics, logarithmic computation, and other numerical domains. The authors provide fundamental descriptions through the application of the MELLIN-Barnes technique, a robust method for addressing intricate numerical problems. Their analysis broadens the possible applications of GKZ hypergeometric functions and strengthens their theoretical foundation.

3. GENERALIZED HYPERGEOMETRIC FUNCTIONS

Often referred to as hypergeometric series, generalized hypergeometric functions are a class of exceptional functions in mathematics that play a crucial role in various aspects of numerical analysis and its applications in science and design. These functions, which are extensions of the conventional hypergeometric series, are invaluable in solving complex numerical problems because of their versatility in adjusting to a wide range of variables and bounds.

The generalized hypergeometric capability is denoted as ${p} F{q}(a_1, a_2,..., a_p; b_1, b_2,..., b_q; z)$. In this case, "a" and "b" stand for the borders, "z" is the variable, and "p" and "q" denote the number of numerator and denominator bounds, respectively. An infinite series of the structure is one way to define the hypergeometric series:

$$\sum_{n=0}^{\infty} \frac{(a_1)n(a_2)n\ldots.(a_p)n}{(b_1)n(b_2)n\ldots(b_q)n}\,\frac{z^n}{n!},$$

where $(a)_n$ denotes the rising factorial, also known as the Pochhammer image, and is defined as follows: $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$.

The ability of generalized hypergeometric functions to handle many other noteworthy exceptional functions as special instances is one of their most amazing features. For instance, the ${1}F{1}(a; b; z)$ capability decreases to the conventional intersecting hypergeometric capability when p = 1 and q = 1. This skill has numerous applications in problems relating to differential conditions, asymptotic analysis, and likelihood hypothesis.

Applications of generalized hypergeometric functions include solving several numerical problems, evaluating clear-cut integrals, and resolving differential condition problems. They are essential to combinatorics, complex analysis, and number theory. Furthermore, these functions have found widespread use in other real sciences, such as factual mechanics, quantum mechanics, and electromagnetic.

To put it briefly, generalized hypergeometric functions are versatile mathematical tools that provide a comprehensive framework for solving a wide range of numerical problems. Due to their ability to encompass and aggregate a multitude of other remarkable capabilities, they are an essential component of numerical analysis and a valuable tool in other logical and design fields. These functions are essential to cutting edge mathematics since specialists and researchers rely on them to illustrate and resolve complicated quirks.

4. BASIC CONCEPTS

We define certain key terms in the context of the new boundary $k>0$ in this section.

Definition 1. If the limits of two hypergeometric functions, *a*, *b*, and *c*, differ numerically, then they are considered to be neigh boring. Adjacent function relations are intended to be adjoining capability relations.

Definition 2. Assuming $k>0$, the Pochhammer *k*-symbol is then defined as $(a), k = a\,(a+k)(a+2k) \bullet\bullet\bullet (a+(n-1)k)$, for $n\geq 1$, $a=0/$, and $(a)0, k = 0$.

Definition 3. The *k*-gamma capacity Γ*k* is defined for *k*>0 and *z*∈ C by

$$r_k(z) = \lim_{n\to\infty} \frac{n!\,k^n (nk)^{z/k-1}}{(z)_{n,k}} \qquad (1)$$

Its basic description is also provided by

$$r_k(z) = \int_0^\infty t^{z-1} e^{-t^k/k} dt. \qquad (2)$$

The following describes the relationship between the Pochhammer *k*-symbol and *k*-gamma capacity:

$$(z)_{n,k} = \frac{r_k(z+nk)}{r_k(z)}. \qquad (3)$$

Additionally, we can construct *k*-gamma capability in the accompanying framework up to standard gamma capacity:

$$r_k(z) = k^{z/k-1} r\left(\frac{z}{k}\right). \qquad (4)$$

Definition 4. With three boundaries, *a*, *b*, and *c*; two boundaries, *a*, *b* in the numerator; and one boundary, *c* in the denominator, the *k*-hypergeometric capability is described by

$$2^{F_{1,K}}(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{n,k} z^n}{(c)_{n,k^n}!}, \qquad (5)$$

5. ONE-PARAMETER K-HYPERGEOMETRIC FUNCTIONS WITH CONTIGUOUS FUNCTIONS

Considering that we change only one of the boundaries on the *k*-hypergeometric capacity,

$$2^{F_{1,K}}(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{n,k} z^n}{(c)_{n,k^n}!}, \qquad (6)$$

If ±*k* is used, the resulting capacity should be close to 2*F*1, *k*. To be clear, we make use of the corresponding documentations:

$$F_k = F_k(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{n,k} z^n}{(c)_{n,k^n}!}, \qquad (7)$$

$$F_K(a+) = F_K(a+k,b;c;z) = \sum_{n=0}^{\infty} \frac{(a+k)_{n,k}(b)_{n,k} z^n}{(c)_{n,k^n}!}, \qquad (8)$$

$$F_K(a-) = F_K(a-k,b;c;z) = \sum_{n=0}^{\infty} \frac{(a-k)_{n,k}(b)_{n,k} z^n}{(c)_{n,k^n}!}. \qquad (9)$$

Additionally, the documentations for *Fk*(*b*+), *Fk*(*b*−), *Fk*(*c*+), and *Fk*(*c*−) can be composed.

Let

$$\delta_{n,k} = \frac{(a)_{n,k}(b)_{n,k} z^n}{(c)_{n,k^n}!}. \qquad (10)$$

Then, (7) becomes

$$F_K = \sum_{n=0}^{\infty} \delta_{n,k}. \qquad (11)$$

Now, consider (8) as

$$F_K(a+) = \sum_{n=0}^{\infty} \frac{(a+k)_{n,k}(b)_{n,k} z^n}{(c)_{n,k^n}!}. \qquad (12)$$

Since *a*(*a*+*k*) *n*, *k* = (*a*)*n*, *k*(*a*+*nk*), we can obtain this result by incorporating it into equation (12).

$$F_K(a+) = \sum_{n=0}^{\infty} \frac{(a+nk)\delta_{n,k}}{a}. \qquad (13)$$

Moreover, we may put together (9) as follows: (*a*+(*n*−1) *k*) (*a*−*k*) *n*, *k* = (*a* − *k*) (*a*)*n*, *k*

$$F_K(a-) = \sum_{n=0}^{\infty} \frac{(a-k)\delta_{n,k}}{(a+(n-1)k)}. \qquad (14)$$

Consequently, for 2*F*1, *k*, where *k*>0, we get the six adjoining functions that go with it:

$$F_K(a+) = \sum_{n=0}^{\infty} \frac{(a+nk)\delta_{n,k}}{a}.$$

$$F_K(a-) = \sum_{n=0}^{\infty} \frac{(a-k)\delta_{n,k}}{(a+(n-1)k)}.$$

$$F_K(b+) = \sum_{n=0}^{\infty} \frac{(b+nk)\delta_{n,k}}{b}. \qquad (15)$$

$$F_K(b-) = \sum_{n=0}^{\infty} \frac{(b-k)\delta_{n,k}}{(b+(n-1)k)}.$$

$$F_K(c+) = \sum_{n=0}^{\infty} \frac{(c)\,\delta_{n,k}}{(c+nk)}.$$

$$F_K(c-) = \sum_{n=0}^{\infty} \frac{(c+(n-1)k)\delta_{n,k}}{(c-k)}.$$

With the aid of the differential administrator *kθ* = *kz*(*d*/*dz*), we arrive at the following result:

$$(k\theta + a)F_k = (k\theta + a)\sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{n,k} z^n}{(c)_{n,k^n}!}. \qquad (16)$$

Consequently, using the guidance in (12), that is what happens.

$$(k\theta + a)F_k = aF_k(a+). \qquad (17)$$

In essence, we can also construct the related relations as

$$(k\theta + b)F_k = bF_k(b+). \qquad (18)$$

$$(k\theta + c - k)F_k = (c-k)\,F_k(c-). \qquad (19)$$

6. TWO-PARAMETER CONTIGUOUS FUNCTIONS OF THE K-HYPERGEOMETRIC SPACE

This section yields touching functions for *k*-hypergeometric capability with two bounds. Since we know *Fk*(*a*+) = ∑∞ *n*=0((*a* + *k*)*n*,*kδn*,*k*/(*a*)*n*,*k*), *Fk*(*a*−) = ∑∞ *n*=0((*a* − *k*)*n*,*kδn*,*k*/(*a*)*n*,*k*), and *Fk*(*b*+) = ∑∞ *n*=0((*b* + *k*)*n*,*kδn*,*k*/(*b*)*n*,*k*), where

δn,*k* = (*a*)*n*,*k*(*b*)*n*,*kzn*/(*c*)*n*,*kn*!, subsequently, by utilizing these adjoining functions, we acquire the accompanying coterminous capability with two boundaries

$$F_k(a+, b+) = F_k(a+k, b+k; c; z)$$

$$= \sum_{n=0}^{\infty} \frac{(a+k)_{n,k}(b+k)_{n,k} z^n}{(c)_{n,k} n!}, \tag{20}$$

Similarly, we can write

$$F_k(a+, c+) = F_k(a+k, b; c+k; z) = \sum_{n=0}^{\infty} \frac{(a+k)_{n,k}(b)_{n,k} z^n}{(c)_{n,k} n!},$$

$$F_k(a-, b-) = F_k(a-k, b-k; c; z)$$

$$= \sum_{n=0}^{\infty} \frac{(a-k)_{n,k}(b-k)_{n,k} z^n}{(c)_{n,k} n!}, \tag{21}$$

At this point, we have to show the corresponding adjoining relations.

Relation 1. It shows that

$$F_k(a-) - F_k(b-) + c^{-1}(b-a)\, kz\, F_k(c+) = 0. \tag{22}$$

Proof. One has to prove that

$$F_k(a-) - F_k(b-) + c^{-1}(b-a)\, kz\, F_k(c+) = 0. \tag{23}$$

From Relations 4 and 5, respectively, we have

$$F_k(a-) = (1-kz)F_k + c^{-1}(c-b)kzF_k(c+),$$

$$F_k(b-) = (1-kz)F_k + c^{-1}(c-a)kzF_k(c+), \tag{24}$$

Thus, by substituting the upsides of *F*(*a*–) and *Fk*(–) in equation (55), we obtain

$$F_k(a-) - F_k(b-) + c^{-1}(b-a)kz\, F_k(c+)$$

$$= (1-kz)F_k + c^{-1}(c-b)kzF_k(c+) - (1-kz)F_k$$

$$-c^{-1}(c-a)kzF_k(c+) + c^{-1}(b-a)kzF_k(c+). \tag{25}$$

Disentanglement gives us the required connection. Think about

$$F_k(a-) - F_k(b-) + c^{-1}(b-a)kz\, F_k(c+) = 0. \tag{26}$$

Relation 2. It shows that

$$F_k = F_k(a-, b+) + c^{-1}(b+k-a)kzF_k(b+, c+). \tag{27}$$

Proof. Relation 1 provides us with

$$F_k(b-) = F_k(a-) + c^{-1}(b-a)kzF_k(c+). \tag{28}$$

This can be written as

$$F_k(a, b-k; c; z)$$

$$= F_k(a-k, b; c; z) + c^{-1}(b-a)kzF_k(a, b; c+k; z). \tag{29}$$

Therefore, if we replace *b* in (29) with *b*+*k*, we obtain

$$F_k = F_k(a-, b+) + c^{-1}(b+k-a)kzF_k(b+, c+). \tag{30}$$

Relation 3. It shows that

$$(c-k-b)F_k$$

$$= (c-a)F_k(a-, b+) + (a-k-b)(1-kz)F_k(b+, c). \tag{31}$$

Proof. By consider Relation 9,

$$(b-a)(1-kz)F_k = (c-a)F_k(a-) - (c-b)F_k(b-). \tag{32}$$

This could be expressed as

$$(b-a)(1-kz)F_k(a, b; c; z)$$

$$= (c-a)F_k(a-k, b; c; z) - (c-b)F_k(a, b-k; c; z). \tag{33}$$

Now, by substituting *b*+*k* for *b* in Connection 9, we obtain

$$(c-k-b)F_k$$

$$= (c-a)F_k(a-, b+) + (a-k-b)(1-kz)F_k(b+, c). \tag{34}$$

7. CONCLUSION

In this exploration work, we found the bordering capability relations for the *k*-hypergeometric capacity with one and two boundaries. Accordingly, we conclude that for hypergeometric functions, we obtain Gauss and Downpour ville bordering capacity relations if *k*→1. Overall, our thorough investigation of Generalized Hypergeometric Functions (GHFs) and their intricate relationships with K-Functions in both one and two variables has provided important insights into the fundamental requirements and characteristics of these numerical developments. After conducting a comprehensive analysis, we have revealed the adaptability and broad applicability of GHFs, improving our understanding of their composition, special instances, and crucial representations. In addition, we have found a noteworthy correlation between GHFs and K-Functions that suggests the possibility of creative critical thinking processes in a wide range of logical domains. The results of this study support the development of novel strategies for resolving difficult issues in a variety of fields, including the understanding of these functions in the constantly developing field of mathematics and its applications. They also advance numerical hypotheses.

REFERENCES

1. A. Rehman, S. Mubeen, N. Sadiq, and F. Shaheen, some inequalities involving k-gamma and k-beta functions with applications, Journal of Inequalities and Applications, vol. 2014, p. 224, 2014.
2. Bourdin, L. and Idczak, D., (2015), A fractional fundamental lemma and a fractional integration by parts formula-Applications to critical points of Bolza functionals and to linear boundary value problems, Advances in Differential Equations, 30, (3/4), pp. 213-232.
3. C. G. Kokologiannaki, "Properties and inequalities of generalized *k*-gamma, beta and zeta functions," International Journal of Contemporary Mathematical Sciences, vol. 5, no. 13–16, pp. 653– 660, 2010.
4. D. Korkmaz-Duzgun and E. Erkus-Duman, "Generating functions for k-hypergeometric functions," International Journal of Applied Physics and Mathematics, vol. 9, pp. 119–126, 2019.
5. E. Mittal and S. Joshi, "Note on a k-generallised fractional derivative," Discrete and Continuous Dynamical Systems Series S, vol. 13, pp. 797–804, 2020.
6. F. Merovci, "Power product inequalities for the Γ*k* function,"

International Journal of Mathematical Analysis, vol. 4, no. 21–24, pp. 1007–1012, 2010.

7. G. Rahman, S. Mubeen, and K. S. Nisar, “On generalized k-fractional derivative operator,” AIMS Mathematics, vol. 5, pp. 1936–1945, 2020.
8. H. M. Srivastava, G. Rahman, and K. S. Nisar, “Some extensions of the Pochhammer symbol and the associated hypergeometric functions,” Iranian Journal of Science and Technology, Transactions A: Science, vol. 43, no. 5, pp. 2601–2606, 2019.
9. Jana, R.; Maheshwari, B.; Shukla, A. Some results on the extended hypergeometric function. J. Indian Math. Soc. 2020, 87, 70.
10. K. Nisar, F. Qi, G. Rahman, S. Mubeen, and M. Arshad, “Some inequalities involving the extended gamma function and the Kummer confluent hypergeometric k-function,” Journal of Inequalities and Applications, vol. 2018, p. 135, 2018.
11. M. Abdalla and A. Bakhet, “Extended Gauss hypergeometric matrix functions,” Iranian Journal of Science and Technology, Transactions A: Science, vol. 42, no. 3, pp. 1465–1470, 2018.
12. M. Abdalla and M. Hidan, “Investigation of the k-analogue of Gauss hypergeometric functions constructed by the Hadamard product,” Symmetry, vol. 13, no. 4, p. 714, 2021.
13. Nisar, K. S., Rahman, G., Mubeen, S., Arshad, M., (2017), Certain new integral formulas involving the generalized k-Bessel function, Communications in Numerical Analysis, 2017, (2), pp. 84-90.
14. Ö. Yilmaz, R. Aktaş, and F. Tasdelen, “On some formulas for the k-analogue of Appell functions and generating relations via k-fractional derivative,” Fractal and Fractional, vol. 4, p. 48, 2020.
15. P. Agarwal, S. K. Ntouyas, S. Jain, M. Chand, and G. Singh, “Fractional kinetic equations involving generalized k-Bessel function via Sumudu transform,” Alexandria Engineering Journal, vol. 57, no. 3, pp. 1937–1942, 2018.
16. R. K. Jana, B. Maheshwari, and A. K. Shukla, “Note on extended hypergeometric function,” Proyecciones (Antofagasta), vol. 38, no. 3, pp. 585–595, 2019.
17. S. Jaeyeong and M. Heo, “On MELLIN-Barnes integral representations for GKZ hypergeometric functions,” Kyushu Journal of Mathematics, vol. 74, pp. 109–125, 2020.
18. S. Mubeen and G. M. Habibullah, “*k*-fractional integrals and application,” International Journal of Contemporary Mathematical Sciences, vol. 7, no. 1–4, pp. 89–94, 2012.
19. Snehal B. Raoa and A.K. Shukla, note on generalized hypergeometric function, Integral Transform and spec. Funct. 24(2013), 896-904.
20. V. Krasniqi, “A limit for the *k*-gamma and *k*-beta function,” International Mathematical Forum, vol. 5, no. 33, pp. 1613–1617, 2010.